DAY SEVEN

System of Particles and Rigid Body

Learning & Revision for the Day

- Centre of Mass
- Rigid Bodies
- Moment of Inertia
- Theorems on Moment of Inertia
- Moment of Inertia of Some Geometrical Oojects

Centre of Mass

Centre of mass of a system (body) is a point that moves when external forces are applied on the body as though all the mass were concentrated at that point and all external forces were applied there.

Centre of Mass of Two Particle System

Centre of mass of a two particles system consisting of two particles of masses m_1, m_2 and respective position vectors $\mathbf{r}_1, \mathbf{r}_2$ is given by

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

If $m_1 = m_2 = m$ (say), then $\mathbf{r}_{CM} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}$

Centre of Mass of *n*-Particle System

Centre of mass of $\mathbf{r}_{CM}$ particles system which consists n-particles of masses $m_1, m_2, \ldots, m_n$ with $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ as their position vectors at a given instant of time is given by

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i\mathbf{r}_i}{M}$$

Cartesian Components of the Centre of Mass

The position vectors $\mathbf{r}_{CM}$ and $\mathbf{r}_i$ are related to their cartesian components by

$$\mathbf{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k} \quad \text{and} \quad \mathbf{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

The cartesian components of $\mathbf{r}_{CM}$ are given by

$$x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{M}, \; y_{CM} = \frac{\sum_{i=1}^{n} m_i y_i}{M} \text{ and } z_{CM} = \frac{\sum_{i=1}^{n} m_i z_i}{M}$$

Motion of Centre of Mass

The position vector $\mathbf{r}_{CM}$ of the centre of mass of n particle system is defined by

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + m_3 \ldots m_n}$$

$$= \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + \ldots + m_n\mathbf{r}_n)$$

$$\frac{d\mathbf{r}_{CM}}{dt} = \frac{1}{M}\left(m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \ldots + m_n\frac{d\mathbf{r}}{dt}\right)$$

Velocity of centre of mass $\mathbf{v}_{CM} = \frac{1}{M}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots m_n\mathbf{v}_n)$

$$\mathbf{v}_{CM} = \frac{\sum_{i=1}^{n} m_i v_i}{M}$$

- Similarly, **acceleration of centre of mass** is given by

$$\mathbf{a}_{CM} = \frac{\sum_{i=1}^{n} m_i \mathbf{a}_i}{M}$$

- From Newton's second law of motion,

$$M\mathbf{a}_{CM} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots \mathbf{F}_n \Rightarrow M\mathbf{a}_{CM} = \mathbf{F}_{Ext}$$

For an isolated system, if external force on the body is zero.

$$\mathbf{F} = M\mathbf{a}_{CM} = M\frac{d}{dt}(\mathbf{v}_{CM}) = 0 \Rightarrow \mathbf{v}_{CM} = \text{constant}$$

i.e. Centre of mass of an isolated system moves with uniform velocity along a straight line path and momentum remain conserved.

NOTE
- If some mass or area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formula

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 - m_2} \text{ or } \mathbf{r}_{CM} = \frac{A_1\mathbf{r}_1 - A_2\mathbf{r}_2}{A_1 - A_2}$$

Momentum Conservation

Let us consider a system of particles of masses $m_1, m_2 \ldots m_n$ are respective velocities $\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_n$. The total linear momentum of the system would be the vector sum of the momentum of the individual particles.

i.e. $p = p_1 + p_2 + p_3 \ldots p_n = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots .. + m_n\mathbf{v}_n$

Velocity of centre of mass of a system

$$\mathbf{v}_{CM} = \frac{1}{M}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots\ldots + m_n\mathbf{v}_n)$$

where, M is the total mass of the system, therefore

$$\mathbf{p} = M\mathbf{v}_{CM}$$

Thus, total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

Again $\frac{d\mathbf{p}}{dt} = M\frac{d\mathbf{v}}{dt} = M\mathbf{a} = F_{ext}$

If $F_{ext} = 0$, then $\frac{dp}{dt} = 0$, i.e. $p = \text{constant}$

If external force of a system is zero, then momentum of system of particle remain constant.

Rigid Bodies

A rigid body is defined as that body which does not undergo any change in shape and volume when external forces are applied on it. When a force is applied on a rigid body, the distance between any two particles of the body will remain unchanged, however, larger the forces may be.

Coordinates of centre of mass of a rigid body are

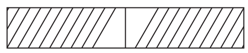
$$X_{CM} = \frac{1}{M}\int x\,dm,$$

$$Y_{CM} = \frac{1}{M}\int y\,dm$$

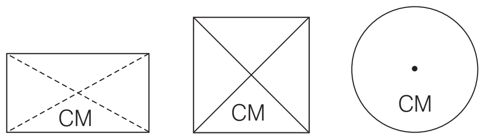
and $$Z_{CM} = \frac{1}{M}\int z\,dm$$

Centre of Mass of Some Rigid Bodies

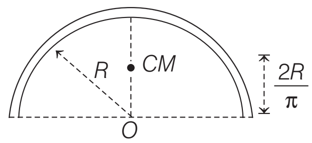
- The centre of mass of a uniform rod is located at its mid-point.



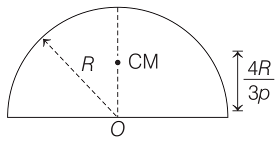
- Centre of mass of a uniform rectangular, square or circular plate lies at its centre.



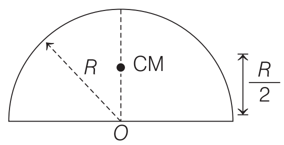
- Centre of mass of a uniform semi-circular ring lies at a distance of $h = \frac{2R}{\pi}$ from its centre, on the axis of symmetry, where R is the radius of the ring.



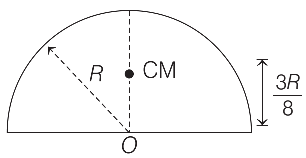
- Centre of mass of a uniform semi-circular disc of radius R lies at a distance of $h = \frac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in figure.



- Centre of mass of a hemispherical shell of radius R lies at a distance of $h = \frac{R}{2}$ from its centre on the axis of symmetry as shown in figure.



- Centre of mass of a solid hemisphere of radius R lies at a distance of $h = \frac{3R}{8}$ from its centre on the axis of symmetry.



Moment of Inertia

Moment of inertia of a rotating body is its property to oppose any change in its state of uniform rotation.

If in a given rotational system particles of masses $m_1, m_2, m_3, \ldots$ be situated at normal distances $r_1, r_2, r_3, \ldots$ from the axis of rotation, then moment of inertia of the system about the axis of rotation is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots = \Sigma mr^2$$

For a rigid body having continuous mass distribution

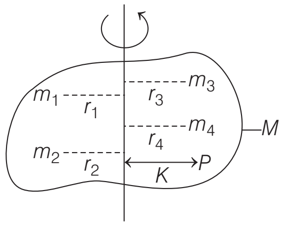
$$I = \int dm\, r^2$$

SI unit of moment of inertia is kg m^2. It is neither a scalar nor a vector i.e. it is a tensor.

Radius of Gyration

Radius of gyration of a given body about a given axis of rotation is the normal distance of a point from the axis, where if whole mass of the body is placed, then its moment of inertia will be exactly same as it has with its actual distribution of mass. Thus, radius of gyration

$$K = \sqrt{\frac{I}{M}} \quad \text{or} \quad K = \left[\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}\right]^{1/2}$$



Axis of rotation

SI unit of radius of gyration is metre.

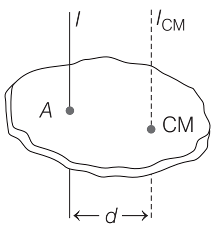
Radius of gyration depends upon shape and size of the body, position and configuration of the axis of rotation and also on distribution of mass of body w.r.t. axis of rotation.

Theorems on Moment of Inertia

There are two theorems based on moment of inertia are given below:

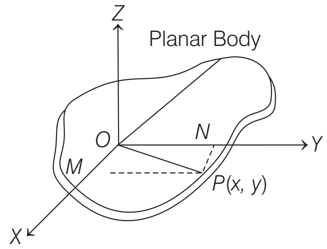
1. Theorem of Parallel Axes

Moment of inertia of a body about a given axis I is equal to the sum of moment of inertia of the body about a parallel axis passing through its centre of mass I_{CM} and the product of mass of body (M) and square of normal distance between the two axes. Mathematically, $I = I_{CM} + Md^2$



2. Theorem of Perpendicular Axes

The sum of moment of inertia of a plane laminar body about two mutually perpendicular axes lying in its plane is equal to its moment of inertia about an axis passing through the point of intersection of these two axes and perpendicular to the plane of laminar body.



If I_x and I_y be moment of inertia of the body about two perpendicular axes in its own plane and I_z be the moment of inertia about an axis passing through point O and perpendicular to the plane of lamina, then

$$I_z = I_x + I_y$$

In theorem of perpendicular axes, the point of intersection of the three axes (x, y and z) may be any point on the plane.

Moment of Inertia of Some Geometrical Objects

- **Uniform Ring of Mass *M* and Radius *R*** About an axis passing through the centre and perpendicular to plane of ring $I = MR^2$. About a diameter $I = \frac{1}{2} MR^2$
- **Uniform Circular Disc of Mass *M* and Radius *R*** About an axis passing through the centre and perpendicular to plane of disc $I = \frac{1}{2} MR^2$. About a diameter $I = \frac{1}{4} MR^2$
- **Thin Uniform Rod of Mass *M* and Length *l*** About an axis passing through its centre and perpendicular to the rod, $I = \frac{1}{12} Ml^2$
- **Uniform Solid Cylinder of Mass *M*, Length *l* and Radius *R*** About its own axis, $I = \frac{1}{2} MR^2$. About an axis passing through its centre and perpendicular to its length $I = M\left[\frac{l^2}{12} + \frac{R^2}{4}\right]$
- **Uniform Solid Sphere** About its diameter $I = \frac{2}{5} MR^2$.

  About its tangent $I = \frac{7}{5} MR^2$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 A body A of mass M while falling vertically downwards under gravity breaks into two parts; a body B of mass $\frac{1}{3}M$ and, a body C of mass $\frac{2}{3}M$. The centre of mass of bodies B and C taken together shifts compared to that of body A towards

(a) depends on height of breaking
(b) does not shift
(c) body C
(d) body B

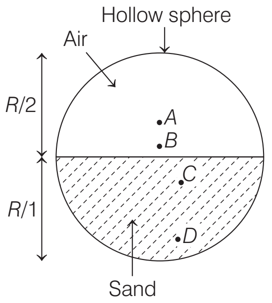
2 Distance of the centre of mass of a solid uniform cone from its vertex is z_0. If the radius of its base is R and its height is h, then z_0 is equal to **→ JEE Main 2015**

(a) $\frac{h^2}{4R}$ (b) $\frac{3h}{4}$ (c) $\frac{5h}{8}$ (d) $\frac{3h^2}{8R}$

3 A circular disc of radius R is removed from a bigger circular disc of radius $2R$, such that the circumference of the discs coincide. The centre of mass of the new disc is αR from the centre of the bigger disc. The value of α is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$

4 Which of the following points is the likely position of the centre of mass of the system as shown in figure?

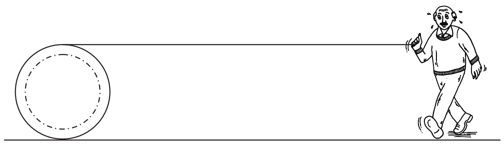


(a) A (b) B (c) C (d) D

5 Consider a two particles system with particles having masses m_1 and m_2. If the first particle is pushed towards the centre of mass through a distance d, by what distance should the second particle be moved, so as to keep the centre of mass at the same position?

(a) $\frac{m_2}{m_1}d$ (b) $\frac{m_1}{m_1+m_2}d$ (c) $-\frac{m_1}{m_2}d$ (d) d

6 A string of negligible thickness is wrapped several times around a cylinder kept on a rough horizontal surface. A man standing at a distance l_m from the cylinder holds one end of the string and pulls the cylinder towards him. There is no slipping anywhere. The length in (m) of the string passed through the hand of the man while the cylinder reached his hands is of



(a) $\frac{l}{2}$ (b) l (c) $2l$ (d) $\frac{3l}{2}$

7 A wheel has mass of the rim 1 kg, having 50 spokes each of mass 5 g. The radius of the wheel is 40 cm. The moment of inertia

(a) 0.273 kg-m^2 (b) 1.73 kg-m^2
(c) 0.173 kg-m^2 (d) 2.73 kg-m^2

8 The surface density of a circular disc of radius a depends on the distance as $\rho(r) = A + Br$. The moment of inertia about the line perpendicular to the plane of the disc is

(a) $\pi a^4\left(\frac{A}{2}+\frac{2a}{5}B\right)$ (b) $\pi a^4\left(\frac{A}{2}+\frac{2B}{5}\right)$
(c) $2\pi a^3\left(\frac{A}{2}+\frac{Ba}{5}\right)$ (d) None of these

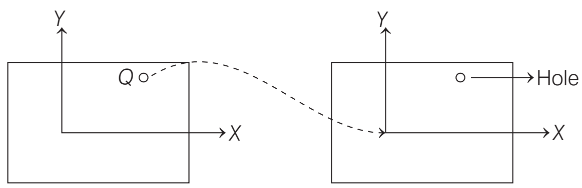
9 Four point masses, each of value m, are placed at the corners of a square $ABCD$ of side l. The moment of inertia of this system about an axis passing through A and parallel to BD is

(a) $2ml^2$ (b) $\sqrt{3}ml^2$ (c) $3ml^2$ (d) ml^2

10 The ratio of the radii of gyration of a circular disc and a circular ring of the same radii about a tangential axis perpendicular to plane of disc or ring is

(a) 1 : 2 (b) $\sqrt{5}:\sqrt{6}$
(c) 2 : 3 (d) $\frac{\sqrt{3}}{2}$

11 A uniform square plate has a small piece Q of an irregular shape removed and glued to the centre of the plate leaving a hole behind. The moment of inertia about the z-axis, then

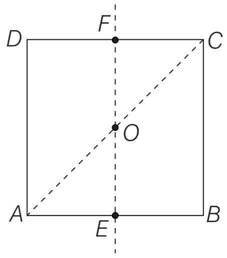


(a) increased
(b) decreased
(c) same
(d) changed in unpredicted manner

12 Consider a uniform square plate of side a and mass m. The moment of inertia of this plate about an axis perpendicular to its plane and passing through one of its corners is

(a) $\frac{5}{6}ma^2$ (b) $\frac{1}{12}ma^2$ (c) $\frac{7}{12}ma^2$ (d) $\frac{2}{3}ma^2$

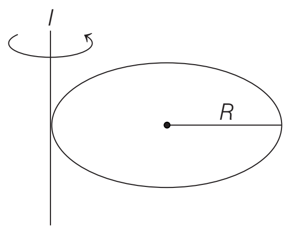
13 For the given uniform square lamina $ABCD$, whose centre is O.



(a) $\sqrt{2}\,I_{AC} = I_{EF}$ (b) $I_{AD} = 3I_{EF}$

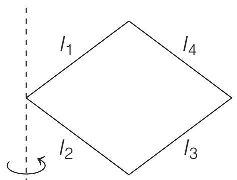
(c) $I_{AC} = I_{EF}$ (d) $I_{AC} = \sqrt{2}\,I_{EF}$

14 A solid sphere of radius R has moment of inertia I about its geometrical axis. It is melted into a disc of radius r and thickness t. If it's moment of inertia about the tangential axis (which is perpendicular to plane of the disc), is also equal to I, then the value of r is equal to



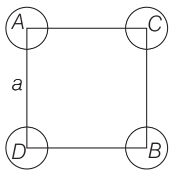
(a) $\frac{2}{\sqrt{15}}R$ (b) $\frac{2}{\sqrt{5}}R$ (c) $\frac{3}{\sqrt{15}}R$ (d) $\frac{\sqrt{3}}{\sqrt{15}}R$

15 The moment of inertia of a system of four rods each of length l and mass m about the axis shown is



(a) $\frac{2}{3}ml^2$ (b) $2\,ml^2$ (c) $3\,ml^2$ (d) $\frac{8}{3}ml^2$

16 Four solid spheres each of mass m and radius r are located with their centres on four corners of a squares $ACBD$ of side a as shown in the figure.



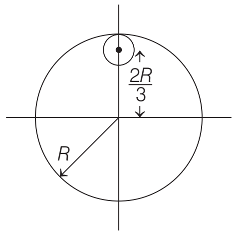
The moment of inertia of the system of four spheres about diagonal AB is

(a) $\frac{m}{5}(8r^2 + 5a^2)$ (b) $\frac{m}{5}(7r^2 + 4a^2)$

(c) $\frac{m}{5}(5r^2 + 8a^2)$ (d) $\frac{m}{5}(3r^2 + 5a^2)$

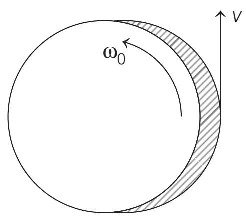
17 From a uniform circular disc of radius R and mass $9\,M$, a small disc of radius $\frac{R}{3}$ is removed as shown in the figure.

The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through centre of disc is **→ JEE Main 2018**



(a) $4MR^2$ (b) $\frac{40}{9}MR^2$ (c) $10MR^2$ (d) $\frac{37}{9}MR^2$

18 A child with mass m is standing at the edge of a merry-go-round having moment of inertia I, radius R and initial angular velocity ω_0 as shown in the figure.



The child jumps off the edge of the merry-go-round with tangential velocity v w.r.t. ground. The new angular velocity of the merry-go-round is

(a) $\left(\frac{I\omega_0^2 - mv^2}{I}\right)^{\frac{1}{2}}$ (b) $\left(\frac{(I + mR^2)\omega_0^2 - mv^2}{I}\right)^{\frac{1}{2}}$

(c) $\left(\frac{I\omega_0 - mvR}{I}\right)$ (d) $\left[\frac{(I + mR^2)\,\omega_0 - mvR}{I}\right]$

Direction (Q. Nos. 19-20) *are the Assertion and Reason type. Each of the these contains two Statements; Assertion and Reason. Each of these question also has four alternative choice,only one of which is correct. You have to select the correct choices from the codes (a),(b), (c) and (d) given below:*

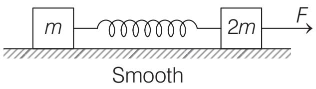
(a) If both Assertion and Reason are true and the Reason is correct explanation of the Assertion

(b) If both Assertion and Reason are true but Reason is not correct explanation of the Assertion

(c) If Asserion is true but Reason is false

(d) If both Assertion and Reason are false

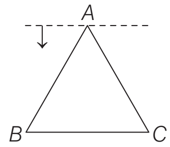
19 **Assertion** (A) A constant force F is applied on the two blocks and one spring system as shown in the figure. Velocity of centre of mass increases linearly with time.



Reason (R) Acceleration of centre of mass is constant.

20 **Assertion** (A) There is a triangular plate as shown in the figure A dotted axis is lying in the plane of the slab. As the axis is moved downwards, moment of inertia of the slab will first decreases, then increases.

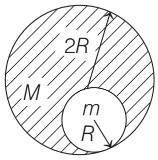
Reason (R) Axis is first moving towards its centre of mass, then it is receding from it.



DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

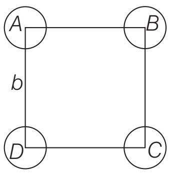
1 Mass of bigger disc having radius $2R$ is M. A disc of radius R is cut from bigger disc as shown in figure. Moment of inertia of disc about an axis passing through periphery and perpendicular to plane is



(a) $\frac{27\,MR^2}{8}$ (b) $\frac{29\,MR^2}{8}$

(c) 3.5 MR (d) $2MR^2$

2 Four spheres of diameter $2a$ and mass M are placed with their centres on the four corners of a square of side b. Then the moment of inertia of the system about an axis along one of the sides the square is



(a) $\frac{4}{5}Ma^2 + 2Mb^2$ (b) $\frac{8}{5}Ma^2 + 2Mb^2$

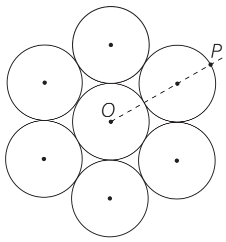
(c) $\frac{8}{5}Ma^2$ (d) $\frac{4}{5}Ma^2 + 4Mb^2$

3 From a solid sphere of mass M and radius R, a cube of maximum possible volume is cut. Moment of inertia of cube about an axis passing through its centre and perpendicular to one of its faces is ➞ **JEE Main 2015**

(a) $\frac{MR^2}{32\sqrt{2}\pi}$ (b) $\frac{MR^2}{16\sqrt{2}\pi}$

(c) $\frac{4MR^2}{9\sqrt{3}\pi}$ (d) $\frac{4MR^2}{3\sqrt{3}\pi}$

4 Seven identical circular planar discs, each of mass M and radius R are welded symmetrically as shown in the figure. The moment of inertia of the arrangement about the axis normal to the plane and passing through the point P is



➞ **JEE Main 2018**

(a) $\frac{19}{2}MR^2$ (b) $\frac{55}{2}MR^2$

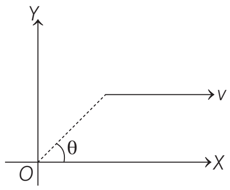
(c) $\frac{73}{2}MR^2$ (d) $\frac{181}{2}MR^2$

5 The moment of inertia of a uniform cylinder of length l and radius R about its perpendicular bisector is I. What is the ratio l/R such that the moment of inertia is minimum? ➞ **JEE Main 2017 (Offline)**

(a) $\frac{\sqrt{3}}{2}$ (b) 1

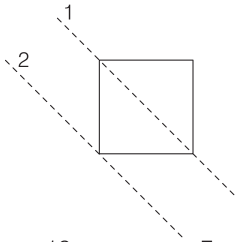
(c) $\frac{3}{\sqrt{2}}$ (d) $\sqrt{\frac{3}{2}}$

6 A particle moves parallel to X-axis with constant velocity v as shown in the figure. The angular velocity of the particle about the origin O is



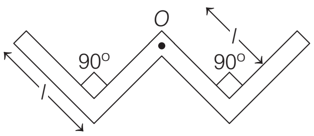
(a) remains constant (b) continuously increasing
(c) continuously decreasing (d) oscillates

7 Let I_1 and I_2 be the moment of inertia of a uniform square plate about an axis as shown in the figure. Then, the ratio $I_1 : I_2$ is



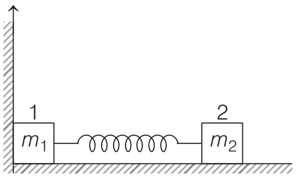
(a) $1 : \frac{1}{7}$ (b) $1 : \frac{12}{7}$ (c) $1 : \frac{7}{12}$ (d) $1 : 7$

8 A thin rod of length $4l$, mass $4m$ is bent at the points as shown in the figure. What is the moment of inertia of the rod about the axis passing through O and perpendicular to the plane of the paper?



(a) $\frac{ml^2}{3}$ (b) $\frac{10\,ml^2}{3}$ (c) $\frac{ml^2}{12}$ (d) $\frac{ml^2}{24}$

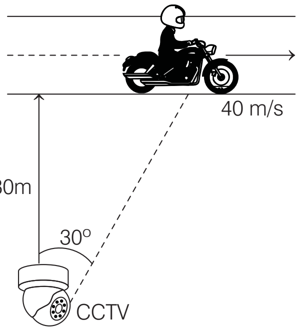
9 Two bars of masses m_1 and m_2 connected by a weightless spring of stiffness k, rest on a smooth horizontal plane. Bar 2 is shifted by a small distance x_0 to the left and released. The velocity of the centre of mass of the system when bar 1 breaks off the wall is



(a) $\frac{\sqrt{km_2}}{m_1 + m_2}$ (b) $\frac{x_0}{m_1 + m_2}\sqrt{km_2}$

(c) $x_0 k\left(\frac{m_1 + m_2}{m_2}\right)$ (d) $x_0 \frac{\sqrt{km_1}}{m_1 + m_2}$

10 A racing bike is travelling along a straight track at a constant velocity of 40 m/s. A fixed CCTV camera is recording the event as shown in the figure. In order to keep the bike in view, in the position shown, the angular velocity of the CCTV camera should be



(a) 3 rad/s
(b) 2 rad/s
(c) 1 red/s
(d) 4 rad/s

11 A rod of length L is placed along the X-axis between $x = 0$ and $x = L$. The linear mass density (mass/length) ρ of the rod varies with the distance x from the origin as $\rho = a + bx$. Here, a and b are constants. The position of centre of mass of the rod is

(a) $\left[\frac{3aL + 2bL^2}{6a + 3bL}, 0, 0\right]$ (b) $\left[\frac{6aL + 3bL^2}{3a + 2bL}, 0, 0\right]$

(c) $\left[\frac{aL + bL^2}{2a + 3bL}, 0, 0\right]$ (d) None of these

ANSWERS

SESSION 1	**1** (b)	**2** (b)	**3** (a)	**4** (c)	**5** (c)	**6** (c)	**7** (c)	**8** (a)	**9** (c)	**10** (d)
	11 (b)	**12** (d)	**13** (c)	**14** (a)	**15** (d)	**16** (a)	**17** (a)	**18** (d)	**19** (a)	**20** (a)
SESSION 2	**1** (b)	**2** (b)	**3** (c)	**4** (d)	**5** (d)	**6** (c)	**7** (d)	**8** (b)	**9** (a)	**10** (c)
	11 (a)									

Hints and Explanations

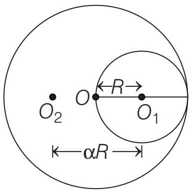
SESSION 1

1 The position of centre of mass remains unaffected because breaking of mass into two parts is due to internal forces.

2 We know that centre of mass of a uniform solid cone of height (h) is at height $\frac{h}{4}$ from base, therefore

As, $h - z_0 = \frac{h}{4}$ or $z_0 = h - \frac{h}{4} = \frac{3h}{4}$

3 In this question, distance of centre of mass of new disc from the centre of mass of remaining disc is αR.



Mass of original disc $= M$

Mass of disc removed

$$= \frac{M}{\pi(2R)^2} \times \pi R^2 = \frac{M}{4}$$

Mass of remaining disc

$$= M - \frac{M}{4} = \frac{3M}{4}$$

$$\therefore \; -\frac{3M}{4}\alpha R + \frac{M}{4} R = 0$$

$$\Rightarrow \quad \alpha = \frac{1}{3}$$

Note *In this question, the given distance must be αR for real approach to the solution.*

4 Centre of mass of a system lies towards that part of the system, having bigger mass. In the given diagram, lower part is heavier, hence CM of the system lies below the horizontal diameter.

Hence, (c) is correct option.

5 To keep the centre of mass at the same position, velocity of centre of mass is zero, so

$$\frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = 0$$

where, $\mathbf{v}_1$ and $\mathbf{v}_2$ are velocities of particles 1 and 2 respectively.

$$\Rightarrow \quad m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} = 0$$

$$\left[\because \mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} \text{ and } \mathbf{v}_2 = \frac{d\mathbf{r}_2}{dt}\right]$$

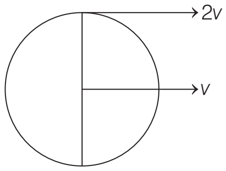
$$\Rightarrow \quad m_1 d\mathbf{r}_1 + m_2 d\mathbf{r}_2 = 0$$

$d\mathbf{r}_1$ and $d\mathbf{r}_2$ represent the change in displacement of particles.

Let second particle has been displaced by distance x.

$$\Rightarrow m_1(d) + m_2(x) = 0 \text{ or } x = -\frac{m_1 d}{m_2}$$

6 If velocity of centre of mass is v, then velocity of contact is 0 and that of the top is $2v$, hence when centre of mass covers a distance l, thread covers a distance $2l$.



7 $I = mr^2 + 50\frac{ml^2}{3}$

$$= 1 \times (0.4)^2 + \frac{50(5 \times 10^{-3})(0.4)^2}{3}$$

$$= 0.16\,(1.083)$$

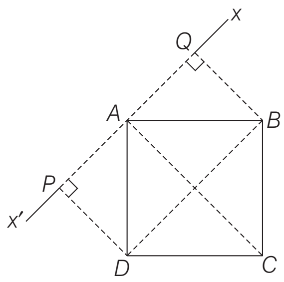
$$= 0.173 \text{ kg-m}^2$$

8 $dm = 2\pi r\, dr\,(\rho) = (A + Br)(2\pi r\, dr)$

$$I = \int_0^a dmr^2 = \frac{\pi A a^4}{2} + \frac{2\pi B a^5}{5}$$

$$= \pi a^4\left(\frac{A}{2} + \frac{2a}{5}B\right)$$

9 The situation is as shown in figure.



$$I_{xx'} = m \times DP^2 + m \times BQ^2 + m \times CA^2$$

$$= m \times 2 \times \left(\frac{\sqrt{2}\,l}{2}\right)^2 + m \times (\sqrt{2}\,l)^2$$

$$= 3\,ml^2$$

10 Radius of gyration,

$$K = \sqrt{\frac{I}{m}}$$

$$\Rightarrow \quad K_{\text{disc}} = \sqrt{\frac{\frac{1}{2}mR^2 + nR^2}{R}} = \sqrt{\frac{3}{2}}R$$

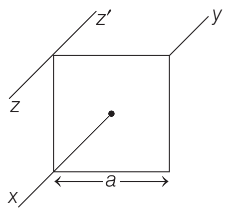
$$K_{\text{ring}} = \sqrt{\frac{mR^2 + mR^2}{m}} = \sqrt{2}R$$

$$\frac{K_{\text{disc}}}{K_{\text{ring}}} = \frac{\sqrt{\frac{3}{2}}}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

11 According to the theorem of perpendicular axes, $I_z = I_x + I_y$ with the hole, I_x and I_y, both decreases (gluing the removed piece at the centre of square plate does not affect I_z).

Hence, I_z decreases overall.

12 Moment of inertia of square plate about xy is $\frac{ma^2}{6}$. Moment of inertia about zz' can be computed using parallel axes theorem



$$I_{zz'} = I_{xy} + m\left(\frac{a}{\sqrt{2}}\right)^2$$

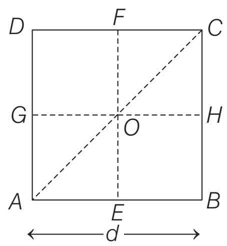
$$= \frac{ma^2}{6} + \frac{ma^2}{2}$$

$$= \frac{2ma^2}{3}$$

13 Let the each side of square lamina is d.

So, $\quad I_{EF} = I_{GH}$ (due to symmetry)

and $\quad I_{AC} = I_{BD}$ (due to symmetry)



Now, according to theorem of perpendicular axes,

$$I_{AC} + I_{BD} = I_0$$

$$\Rightarrow \quad 2I_{AC} = I_0 \quad \text{...(i)}$$

and $\quad I_{EF} + I_{GH} = I_0$

$$\Rightarrow \quad 2I_{EF} = I_0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$I_{AC} = I_{EF}$$

14 $\frac{2}{5}MR^2 = \frac{1}{2}Mr^2 + Mr^2$

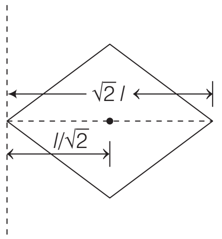
or $\frac{2}{5}MR^2 = \frac{3}{2}Mr^2$

$\therefore \quad r = \frac{2}{\sqrt{15}}R$

15 Consider a square lamina, then

$4m\left(\frac{l^2}{12} + \frac{l^2}{12}\right)$ about

Centre of mass $= \frac{2ml^2}{3}$



Apply perpendicular axis theorem,

$$= \frac{2ml^2}{3} + 4m\left(\frac{l}{\sqrt{2}}\right)^2 = \frac{8}{3}ml^2$$

16 The moment of inertia of the system of four spheres about diagonal AB is

I_{AB} = MI of A about AB + MI of B about AB + MI of C about AB + MI of D about AB

$$= \frac{2}{5}mr^2 + \frac{2}{5}mr^2 + \left(\frac{2}{5}mr^2 + \frac{1}{2}ma^2\right) + \left(\frac{2}{5}mr^2 + \frac{1}{2}ma^2\right)$$

$$= \frac{8}{5}mr^2 + ma^2 = m\left(\frac{8r^2}{5} + a^2\right)$$

$$= \frac{m}{5}(8r^2 + 5a^2)$$

17 Moment of inertia of remaining solid = Moment of inertia of complete solid − Moment of inertia of removed portion

$$\therefore \quad I = \frac{9MR^2}{2} - \left[\frac{M(R/3)^2}{2} + M\left(\frac{2R}{3}\right)^2\right]$$

$\Rightarrow I = 4MR^2$

18 Since, in this condition,

Initial angular momentum = Final angular momentum

$\therefore (I + mR^2)\omega_0 = (mvR) + I\omega'$

or $\quad \omega' = \frac{(I + mR^2)\omega_0 - mvR}{I}$

Hence, (d) is the correct option.

19 Total mass of the system $= m + 2m = 3m$

Force applied on the system is F.

$\therefore \quad a_{CM} = \frac{F}{3m}$ = constant as F is constant

$\therefore \quad v_{CM} = a_{CM} \times t$

or $v_{CM} = \frac{F}{3m} \times t$ or $v_{CM} \propto t$

Hence, both Assertion and Reason are true and the Reason is the correct explanation of Assertion.

20 Moment of inertia $(I) = mr^2$, where r is distance from the axis of rotation to the centre of mass. When dotted axis moved downward (towards centre of mass), r decreases result moment of inertia decrease and when dotted axis cross the centre of mass and moved further downwards then r increases result moment of inertia increases. Hence, both Assertion and Reason are true and the Reason is the correct explanation of Assertion.

SESSION 2

1 Surface density of motional disc is

$$\sigma = \frac{M}{\pi(2R)^2} = \frac{M}{4\pi R^2}$$

Mass of cutting portion is

$$m_1 = \sigma \times \pi R^2 = \frac{M}{4}$$

$$I = I_1 - I_2$$

where,

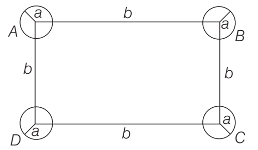
I_1 = Moment of inertia of disc about given axis without cutting portion

I_2 = Moment of inertia due to cutting portion

$$I = \frac{M(2R)^2}{2} + M(2R)^2 - \left[\frac{m_1R^2}{2} + m_1(3R)^2\right]$$

$$= 6MR^2 - \frac{19MR^2}{8} = \frac{29MR^2}{8}$$

2



Moment of inertia of each of the spheres A and D about

$$AD = \frac{2}{5}Ma^2$$

Moment of inertia of each of the spheres B and C about AD

$$= \left(\frac{2}{5}Ma^2 + Mb^2\right)$$

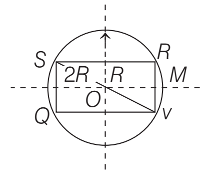
Using theorem of parallel axis, we get

Total moment of inertia

$$= I\left[\frac{2}{5}Ma^2\right] \times 2 + \left[\frac{2}{5}Ma^2 + Mb^2\right] \times 2$$

$$I = \frac{8}{5}Ma^2 + 2Mb^2$$

3 Consider the cross-sectional view of a diametric plane as shown in the adjacent diagram.

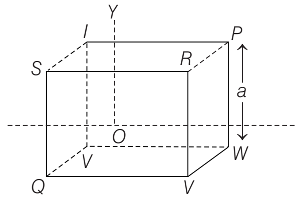


Cross-sectional view of the cube and sphere

Using geometry of the cube

$$PQ = 2R = (\sqrt{3})\,a \text{ or } a = \frac{2R}{\sqrt{3}}$$

Volume density of the solid sphere,



$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3}{3\pi}\left(\frac{M}{R^3}\right)$$

Mass of cube $(m) = (\rho)(a)^3$

$$= \left(\frac{3}{4\pi} \times \frac{M}{R^3}\right)\left[\frac{2R}{\sqrt{3}}\right]^3$$

$$= \frac{3M}{4\pi R^3} \times \frac{8R^3}{3\sqrt{3}} = \frac{2M}{\sqrt{3}\pi}$$

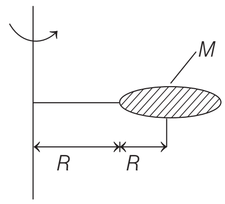
Moment of inertia of the cube about given axis is

$$I_y = \frac{m}{12}(a^2 + a^2) = \frac{ma^2}{6}$$

$$\Rightarrow I_y = \frac{ma^2}{6} = \frac{2M}{\sqrt{3}\pi} \times \frac{1}{6} \times \frac{4R^2}{3} = \frac{4MR^2}{9\sqrt{3}\pi}$$

4 Moment of inertia of an outer disc about the axis through centre is

$$= \frac{MR^2}{2} + M(2R)^2 = MR^2\left(4 + \frac{1}{2}\right) = \frac{9}{2}MR^2$$



For 6 such discs,

Moment of inertia $= 6 \times \frac{9}{2}MR^2$

$= 27MR^2$

So, moment of inertia of system

$$= \frac{MR^2}{2} + 27MR^2 = \frac{55}{2}MR^2$$

Hence, $I_P = \frac{55}{2}MR^2 + (7M \times 9R^2)$

$\Rightarrow \quad I_P = \frac{181}{2}MR^2$

$I_{system} = \frac{181}{2}MR^2$

5 MI of a solid cylinder about its perpendicular bisector of length is

$$I = m\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$$

$$\Rightarrow \quad I = \frac{mR^2}{4} + \frac{ml^2}{12}$$

$$= \frac{m^2}{4\pi\rho l} + \frac{ml^2}{12} \qquad [\because \rho\pi R^2 l = m]$$

For I to be maximum,

$$\frac{dI}{dl} = -\frac{m^2}{4\pi\rho}\left(\frac{1}{l^2}\right) + \frac{ml}{6} = 0$$

$$\Rightarrow \quad \frac{m^2}{4\pi\rho} = \frac{ml^3}{6}$$

Now, putting $m = \rho\pi R^2 l$

$$\therefore \quad l^3 = \frac{3}{2\pi\rho}\cdot\rho\pi R^2 l$$

$$\frac{l^2}{R^2} = \frac{3}{2}$$

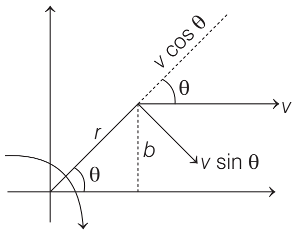
$$\therefore \quad \frac{l}{R} = \sqrt{\frac{3}{2}}$$

6 Since, $\omega = \frac{v}{r}$

But, here

$v = v \sin\theta$

and $\quad r = b/\sin\theta$



$$\therefore \quad \omega = \frac{v\sin\theta}{b/\sin\theta}$$

$$= \frac{v\sin^2\theta}{b}$$

or $\quad \omega \propto \sin^2\theta$

[as v and b are constants]

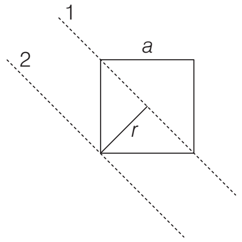
$\therefore$ When particle moves further parallel to the X-axis, θ decreases result angular velocity decreases.

Hence, (c) is the correct option.

7 Here, $I_1 = \frac{ma^2}{12}$ and $I_2 = I_1 + mr^2$

From figure, we get

$$r = \frac{a}{\sqrt{2}}$$

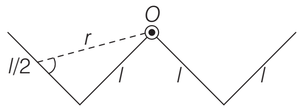


$$\therefore I_2 = \frac{ma^2}{12} + \frac{ma^2}{2} = \frac{7}{12}ma^2$$

Hence, $I_1 : I_2 = 1:7$

Thus, (d) is the correct option.

8 From figure,



$$r = \sqrt{l^2 + \frac{l^2}{4}} = \frac{\sqrt{5}}{2}l$$

Due to symmetry,

$$I = 2\,[I_1 + I_2] = 2\left[\frac{ml^2}{3} + \left(\frac{ml^2}{12} + mr^2\right)\right]$$

$$= 2\left[\frac{ml^3}{3} + \frac{ml^2}{12} + \frac{5ml^2}{4}\right] = \frac{10}{3}ml^2$$

Hence, (b) is the correct option.

9 Here, $\frac{1}{2}kx_0^2 = \frac{1}{2}m_2v_2^2$

$$\therefore \quad v_2 = \sqrt{\frac{k}{m_2}}\,x_0$$

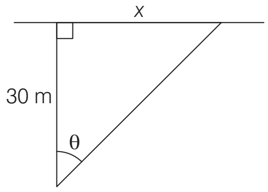
Since, $v_{CM} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$

$$\therefore \quad v_{CM} = \frac{m_1 \times 0 + m_2\sqrt{\frac{k}{m_2}}\,x_0}{m_1 + m_2}$$

$$= \frac{\sqrt{k\,m_2}}{m_1 + m_2}\cdot x_0$$

Hence, (a) is the correct option.

10 Here, $\tan\theta = \frac{x}{30}$ or $x = 30\tan\theta$



Now, differentiate both sides w.r.t. time,

$$\frac{dx}{dt} = 30 \times \sec^2\theta\cdot\frac{d\theta}{dt}$$

But $\frac{dx}{dt} = v_{\text{bike}}$ and $\frac{d\theta}{dt} = W$

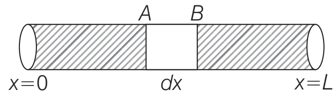
$$\therefore \quad W = \frac{v_{\text{bike}}}{30\sec^2\theta} = \frac{40}{30\sec^2 30°}$$

or $\quad W = 1$ rad/s

Hence, (c) is the correct option.

11 Let dm is the mass of element AB of length dx at distance x

$$\therefore \quad dm = \rho\cdot dx = (a + bx)\cdot dx$$



The centre of mass of the element has coordinates $(x, 0, 0)$.

Therefore, x-coordinate of centre of mass of the rod will be

$$x_{CM} = \frac{\int_0^L x\cdot dm}{\int_0^L dm} = \frac{\int_0^L x\,(a + bx)\,dx}{\int_0^L (a + bx)\,dx}$$

$$= \frac{\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^L}{\left[ax + \frac{bx^2}{2}\right]_0^L}$$

or $\quad x_{CM} = \frac{3aL + 2bL^2}{6a + 3bL}$

$y_{CM} = 0$,

$z_{com} = 0$

$\therefore$ Centre of mass of the rod will be

$$\left[\frac{3aL + 2bL^2}{6a + 3bL}, 0, 0\right]$$

Hence, (a) is the correct option.